Introductio

in Elementarem Functionum Ellipticarum Theoriam,

quam

Cons. Ampliss. Philos. Ord. Lundens.,

Præside

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Pro Laurea

p. p.

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$= 2 . \left(\frac{\alpha}{T^3} + \beta . T\right) d T$, & igitur integrando $- C_{,} . R^2 = \beta T^2 - C \frac{-\alpha}{T^2}$, exsistente $C =$ const. Invenimus igitur $C_{,} R^2 T^2 = \alpha + C . T^2 - \beta T^4$, ideoque, cum $C_{,} R . T = \frac{-dT}{d\varphi}$ sit, erit $\frac{-dT}{d\varphi} = \pm \sqrt{C_{,}(\alpha + CT^2 - \beta T^4)}$. Omnes igitur functiones F ejus indolis, ut

$F(\psi - \varphi) = \frac{F\psi . \mathfrak{d} F\varphi - F\varphi . \mathfrak{d} F\psi}{\alpha + \beta F\varphi^2 . F\psi^2}$ sit, necessario ejusmodi sint, ut

$\mathfrak{d}_{\varphi} F = \pm \sqrt{C_{,}(\alpha + CF^2 - \beta F^4)}$ sit seu $\varphi = \int \frac{dF}{\sqrt{C_{,}(\alpha + CF^2 - \beta F^4)}}$

& præterea evolvendo sit $Fx = C_{,} x + C_{,,} x^2 + C_{,,,} x^3 + \ldots$, ideoque $Fo = o$. Nostræ vero functiones hypercyclicæ ex ejusmodi formula integrali ortum ducunt, & præsertim tangens & uterque sinus ad postremam quoque conditionem quadrant, quare in harum secundum argumentum (sc. aream) transmutationem quam maxime universalem hac via incidimus. Istam vero formulam differentialem evolvendo & integrando, invenitur $\varphi = \frac{F}{\sqrt{C_{,}\alpha}} + D F^3 + \ldots$, hancque seriem invertendo

$F = \varphi . \sqrt{C_{,}\alpha} + D_{,}\varphi^3 + \ldots$, constantibus scilicet D & $D_{,}$ rite determinatis. Ut igitur hæc series cum priori conveniat, necessario sit $\sqrt{C_{,}\alpha} = C_{,}$, seu $C_{,} = \alpha$, atque $C_{,,} = o$; ideoque *theorema* jam inventum ita enuntiari potest: "Si fuerit $\varphi = \int_0^{F\varphi} \frac{dx}{\sqrt{\alpha(\alpha + Cx^2 - \beta x^4)}}$, (habito nempe limite integrationis arbitrario ipsius x ut functione ipsius integralis φ, nempe limitem ipsius $x = F\varphi$),

seu $\mathfrak{d}_{\varphi} F\varphi = \sqrt{\alpha(\alpha + CF\varphi^2 - \beta F\varphi^4)}$, erit universim

$F(\psi-\varphi)=\frac{F\psi\,.\,\mathfrak{d}F\varphi-F\varphi\,\mathfrak{d}F\psi}{\alpha+\beta\,.\,F\varphi^2\,.\,F\psi^2}$; & vice versa" *). In specie igitur, quia in hypercyclicis est

$$\mathfrak{d}T\varphi=\sqrt{(a^2+T\varphi^2)(b^2+T\varphi^2)}=\sqrt{\widehat{ab}^2+\overline{a^2+b^2}\,T\varphi^2+T\varphi^4}=\mathfrak{S}\varphi\,.\,\dot{\mathfrak{S}}\varphi\,.$$

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*) (Patet vero, $\varphi=-\chi$ poni posse, quo casu $F\varphi$ in $-F\chi$ mutatur). Ut theorema omnibus numeris absolutum constet, cæteræ ipsius r^n coefficientes conferendæ essent; quod vero opus, cum infiniti sit ordinis, hunc in modum evitari potest. Ponamus $F(\varphi+\psi)=\frac{p}{q}$, et differentiemus secundum φ & ψ, eritque, si $F'\varphi=\mathfrak{d}F\varphi=\frac{dF\varphi}{d\varphi}$, $F(\varphi+\psi)=\underset{\varphi}{\mathfrak{d}}\left(\frac{p}{q}\right)=\underset{\psi}{\mathfrak{d}}\left(\frac{p}{q}\right)$, ideoque $q\underset{\varphi}{\mathfrak{d}}p-p\underset{\varphi}{\mathfrak{d}}q=q\underset{\psi}{\mathfrak{d}}p-p\underset{\psi}{\mathfrak{d}}q$, seu $\frac{p}{q}=\frac{\underset{\psi}{\mathfrak{d}}p-\underset{\varphi}{\mathfrak{d}}p}{\underset{\psi}{\mathfrak{d}}q-\underset{\varphi}{\mathfrak{d}}q}$. Jam vero in casu nostro est $p=F\varphi\,.\,F'\psi+F\psi\,.\,F'\varphi$, atque $q=\alpha+\beta F\varphi^2.\,F\psi^2$, existente $\underset{\varphi}{\mathfrak{d}}F\varphi=F'\varphi=\sqrt{\alpha\,.\,(\alpha+C\,.\,F\varphi^2-\beta\,F\varphi^4)}$; ideoque $\underset{\varphi}{\mathfrak{d}}p-\underset{\psi}{\mathfrak{d}}p=F\psi\,.\,F''\varphi-F\varphi\,.\,F''\psi$

& $\underset{\varphi}{\mathfrak{d}}q-\underset{\psi}{\mathfrak{d}}q=2\beta F\varphi F\psi\,.\,(F\psi\,.\,F'\varphi-F\varphi\,.\,F'\psi)$. Sit igitur necesse est, $(F\varphi\,F'\psi+F\psi\,F'\varphi)\,.\,2\,\beta\,F\varphi\,F\psi\,.\,(F\psi\,F'\varphi-F\varphi\,F'\psi)$ seu $2\beta F\varphi F\psi\,.\,(F\psi^2\,F'\varphi^2-F\varphi^2\,F'\psi^2)=(\alpha+\beta\,F\varphi^2.\,F\psi^2)\,.\,(F\psi\,.\,F''\varphi-F\varphi\,F''\psi)$. Est vero $F\psi^2\,.\,F'\varphi^2-F\varphi^2\,.\,F'\psi^2=\alpha\,.\,(F\psi^2-F\varphi^2)\,.\,(\alpha+\beta F\varphi^2\,.\,F\psi^2)$, ideoque erit $2\alpha\beta\,F\varphi\,F\psi\,(F\psi^2-F\varphi^2)=F\psi\,F''\varphi-F\varphi.F''\psi$ seu $2\alpha\beta\,(F\psi^2-F\varphi^2)=\frac{F''\varphi}{F\varphi}-\frac{F''\psi}{F\psi}$. Quia vero $\widehat{F\varphi}^2=\alpha\,.\,(\alpha+C\,.\,F\varphi^2-\beta\,F\varphi^4)$, differentiando hanc aquationem $2\,.\,F'\varphi\,.\,F''\varphi=\alpha.F'\varphi\,.\,(2\,CF\varphi-4\beta F\varphi^3)$ erit, & $\frac{F''\varphi}{F\varphi}=\alpha\,.\,(C-2\beta\,F\varphi^2)$, & similiter $\frac{F''\psi}{F\psi}=\alpha\,.\,(C-2\,\beta F\psi^2)$; unde obtinetur $\frac{F''\varphi}{F\varphi}-\frac{F''\psi}{F\psi}=2\,\alpha\beta\,.\,(F\psi^2-F\varphi^2)$, sicuti oportuit. Quare Theorema in textu exhibitum universale bene sibi constat.

erit $T(\psi-\varphi)=\dfrac{T\psi.\mathfrak{d}T\varphi-T\varphi.\mathfrak{d}T\psi}{ab-(ab)^{-1}.\overline{T\varphi}^2.T\psi^2}$. Atque similiter, sumto sinu majori S pro F, $S\overline{\psi-\varphi}=\dfrac{S\psi\mathfrak{d}S\varphi-S\varphi.\mathfrak{d}S\psi}{ac-(ac)^{-1}.\overline{S\varphi}^2.\overline{S\psi}^2}$; quia

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Introducto vero valore ipsius $\mathfrak{d}F\varphi$, hoc ita exprimitur:

$$F(\psi+\varphi)=\frac{F\psi.\sqrt{\alpha(\alpha+CF\varphi^2-\beta F\varphi^4)}+F\varphi.\sqrt{\alpha(\alpha+CF\psi^2-\beta F\psi^4)}}{\alpha+\beta.F\varphi^2.F\psi^2}.$$

Hoc est illud celebratissimum Theorema ab Eulero inventum, diversissima tamen via. Ipse enim ex casu speciali, cujus solutionem exhibuerat Cel. Fagnani, generatim qvidem divinatus est, æquationem hujusmodi differentialem $\dfrac{dx}{\sqrt{4^ox}}=\dfrac{dy}{\sqrt{4^oy}}$ integrali gaudere algebraico, eoque rationali in x et y seccundi vel qvarti gradus. Deinde rem varii variis modis adgressi sunt; inter omnes vero eminet Cel. Abel, qui modum generalem invenit ejusmodi æquationes differentiales, si vel altioris essent gradus, integrandi, eumque præstantissimum. Hic vero profundissimos postulat calculos, ideoque ad tempus omittitur. Rem vero si tironibus exponere velis, ita forsan simplicissime absolves. Ponas functionem algebraicam, ab Eulero jam inventam $\dfrac{x\sqrt{1+Cy^2+y^4}-y\sqrt{1+Cx^2+x^4}}{1-x^2y^2}=z$; & disqviras num detur & qualis sit functio (F), qua si hæc formula afficitur, separantur x & y, & quidem ita, ut sit $Fz=Fx-Fy$. Primum igitur patet, posito $y=x$, fore $z=o$, ideoque $Fo=Fx-Fx=o$. Deinde vero ex Theoremate, quod a Cel. Maclaurin nomen gerit, (posito breviter $\mathfrak{d}^nFx$ loco τοῦ $\dfrac{d^nFx}{1.2..n(dx)^n}$), erit:

$Fx=Fo+x\mathfrak{d}Fo+x^2\mathfrak{d}^2Fo+..=x\mathfrak{d}Fo+x^2\mathfrak{d}^2Fo+.....$ Posito jam $x=y+e$, erit, irrationalibus in denominatorem translatis, $z=$
$\dfrac{x^2(1+Cy^2+y^4)-y^2(1+Cx^2+x^4)}{(1-x^2y^2).(x\sqrt{1+Cy^2+y^4}+y\sqrt{1+Cx^2+x^4})}$, seu, abbreviando per $1-x^2y^2$,
$z=\dfrac{x^2-y^2}{x\sqrt{1+Cy^2+y^4}+y\sqrt{1+Cx^2+x^4}}$, ideoque, substitutione et evolutione rite

$$\mathfrak{d}S\varphi = \sqrt{(a^2 - S\varphi^2)(c^2 - S\varphi^2)} = \mathfrak{T}^c\varphi \cdot \dot{S}^c\varphi \left(= \frac{dS\varphi}{d\varphi} \right).$$

Et tandem, si sinus minor ($\dot{S}$) adhibetur loco ipsius F,

erit $\dot{S}(\psi - \varphi) = \dfrac{\dot{S}\psi\,\mathfrak{d}\dot{S}\varphi - \dot{S}\varphi . \mathfrak{d}\dot{S}\psi}{bc + \frac{1}{bc} . \overline{\dot{S}\varphi}^2 . \overline{\dot{S}\psi}^2}$, ubi $\mathfrak{d}\dot{S}\varphi = \sqrt{(b^2 + \dot{S}\varphi^2).(c^2 - \dot{S}\varphi^2)}$

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instituta, $z = \dfrac{\overline{y+e}^2 - y^2}{(y+e)\sqrt{1 + Cy^2 + y^4} + y\sqrt{1 + C\overline{y+e}^2 + \overline{y+e}^4}}$, seu

$$z = \frac{e}{\sqrt{1 + Cy^2 + y^4}} + e^2 J + e^3 J_{,} \ldots$$

exsistente J (& $J_{,}$ &c.) functione ipsius y, evolutionem persequendo facillima inventu, cujus tamen in præsenti nullus est usus.

Habes igitur

$$F\left(\frac{e}{\sqrt{1 + Cy^2 + y^4}} + e^2 J + . \right) = F\overline{y+e} - Fy,$$

ideoque evolutione et theor. Maclaurineo atque Tayloreo instituta,

$$\frac{e\,\mathfrak{d}Fo}{\sqrt{1 + Cy^2 + y^4}} + e^2 J' + . . = e\,\mathfrak{d}Fy + e^2\mathfrak{d}^2 Fy + \ldots$$

Terminis igitur similibus invicem comparatis, erit $\mathfrak{d}Fy = \dfrac{\mathfrak{d}Fo}{\sqrt{1 + Cy^2 + y^4}}$, ideoque, integrando a limite $= o$, $Fy = \mathfrak{d}Fo. \displaystyle\int_o \frac{dy}{\sqrt{1 + Cy^2 + y^4}}$. Functio igitur, quam quæris, ea ipsa est, cujus derivata prima est $\dfrac{a}{\sqrt{1 + Cy^2 + y^4}}$, exsistente $a = \mathfrak{d}Fo$ quantitate constante arbitraria; ideoque erit vice versa $\displaystyle\int_o \frac{dy}{\sqrt{1 + Cy^2 + y^4}} \left(= \frac{Fy}{a} \right)$ nova quædam Functio ipsius y hac proprietate insignis: $Fz = Fx - Fy$, exsistente $z = \dfrac{x\sqrt{1 + Cy^2 + y^4} - y\sqrt{1 + Cx^2 + x^4}}{1 - x^2 y^2}$;

$= \mathfrak{T}\varphi \cdot S^c\varphi$. Quoniam vero $Fo = o$, ideoque $F'o = \mathfrak{d}Fo = \sqrt{\alpha . \alpha} = \alpha$, atque $\alpha . \beta = + 1$ vel (in formula ad $\dot{S}$) $= - 1$, erit, si $F'\varphi = \mathfrak{d}F\varphi$, universim $F(\psi - \varphi) = \frac{F\psi . F'\varphi - F\varphi . F'\psi}{F'o \mp (F'o)^{-1} . F\varphi^2 . F\psi^2}$, significante F vel tangentem (T) vel alterutrum sinuum ($S, \dot{S}$), (sumto tamen signo $+$ ad formulam ipsius $\dot{S}$); & quoniam hæ (Tangens & uterque Sinus) impares sunt functiones, $F'\varphi \left(\text{seu } \frac{dF\varphi}{d\varphi} \right)$ vero par, erit $F(-\varphi) = -F\varphi$ &

$F'(-\varphi) = F'\varphi$, ideoque $F(\psi + \varphi) = \frac{F\psi . F'\varphi + F\varphi F'\psi}{F'o \mp \frac{1}{F'o} \cdot F\varphi^2 . F\psi^2}$,

seu $F(\varphi + \psi) = \frac{F\varphi F'\psi + F\psi F'\varphi}{\alpha - \beta F\varphi^2 F\psi^2}$. (Cfr. p. 74. *).

Has vero formulas universales jungendo, infinitas speciales elicies: ut

1) $F(\psi + \varphi) + F(\psi - \varphi) = \frac{2F\psi . F'\varphi . F'o}{\overline{F'o}^2 \mp F\varphi^2 . F\psi^2}$,

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ipsaque est sector Hypercyclicus, cujus tangens y. Alio etiam modo problemata istiusmodi functionalia solvere licet: Æquationem $Fx + Fy = Fz$ partialiter differentiando, tumque dividendo, habebitur $\frac{\frac{\mathfrak{d}Fx}{x}}{\frac{\mathfrak{d}Fy}{y}} = \frac{\frac{\mathfrak{d}z}{x}}{\frac{\mathfrak{d}z}{y}}$; tum vero valorem ipsius z introducendo & differentiando, calculo (qui tamen nuper exhibito prolixior est) rite subducendo, obtinetur $\frac{\frac{\mathfrak{d}(Fx)}{x}}{\frac{\mathfrak{d}(Fy)}{y}} = \frac{\sqrt{1 + Cy^2 + y^4}}{\sqrt{1 + Cx^2 + x^4}}$, indeque (ob x & y a se invicem haud pendentes) colligitur $Fx = a\int \frac{dx}{\sqrt{1 + Cx^2 + x^4}}$. Quin etiam hac via eodemque fere modo, quem et initio hujus notæ adhibuimus, commonstrari potest, reliquas coefficientes serierum, in altera solutione adhibitarum, juste inter se constare. At tironibus calculus modo expositus facilior forsan videbitur.

2) $F\overline{\psi+\varphi}\,.\,F\overline{\psi-\varphi} = \dfrac{\overline{F\psi\,.\,F'\varphi}^2 - \overline{F\varphi\,.\,F'\psi}^2}{(F'o \mp \overline{F'o}^1\,.\,F\varphi^2\,F\psi^2)^2}$,

seu, $F\overline{\psi+\varphi}\,.\,F\overline{\psi-\varphi} = \dfrac{(F\psi^2 - F\varphi^2)\,.\,(F'o)^2}{F'o^2 \mp F\psi^2\,.\,F\varphi^2}$,

quia $\overline{F'\varphi}^2 = \alpha^2 + \gamma\,\overline{F\varphi}^2 \pm \overline{F\varphi}^4$, (posito scilicet $\gamma = \alpha C$), ideoque $F\psi^2\,.\,F^1\varphi^2 - F\varphi^2\,.\,F^1\psi^2 = \alpha^2\,.\,(F\psi^2 - F\varphi^2) \pm (F\psi^2\,F\varphi^4 - F\varphi^2\,.\,F\psi^4)$ seu $= (F\psi^2 - F\varphi^2)\,.\,(F^1o^2 \mp F\psi^2\,F\varphi^2)$.

3) $F\overline{2\varphi} = \dfrac{2\,F\varphi\,.\,F^1\varphi\,.\,F^1o}{(F^1o)^2 \mp F\varphi^4}$, nempe speciatim

a) $T\overline{2\varphi} = \dfrac{2\,T^1o\,.\,T\varphi\,.\,T^1\varphi}{(T^1o)^2 - T\varphi^4}$ seu $= \dfrac{2\,ab\,.\,T\varphi\,.\,\mathfrak{S}\varphi\,.\,\dot{\mathfrak{S}}\varphi}{(ab)^2 - T\varphi^4}$,

b) $S\overline{2\varphi} = \dfrac{2\,S^1o\,.\,S\varphi\,.\,S^1\varphi}{(S^1o)^2 - S\varphi^4}$ seu $S\,2\varphi = \dfrac{2\,ac\,.\,S\varphi\,.\,\dot{S}^c\varphi\,.\,\mathfrak{T}^c\varphi}{(ac)^2 - (S\varphi)^4}$,

& c) $\dot{S}\overline{2\varphi} = \dfrac{2\,\dot{S}^1o\,.\,\dot{S}\varphi\,.\,\dot{S}^1\varphi}{(\dot{S}^1o)^2 + (\dot{S}\varphi)^4} = \dfrac{2\,bc\,.\,\dot{S}\varphi\,.\,S^c\varphi\,.\,\mathfrak{T}\varphi}{(bc)^2 + (\dot{S}\varphi)^4}$. Et si $\varphi = A$ fuerit, ideoque $T\varphi = \infty = \mathfrak{S}\varphi$, $S\varphi = c = \dot{S}\varphi$, $\dot{S}^c\varphi = o$ (vide p. 21), erit $T2\varphi = T2A = \dfrac{-2\,ab\,.\,T\varphi^3}{T\varphi^4} = o$, $S\,2A = o = \dot{S}\,2A$. Hinc vero facile evincitur, universim $S\,2nA = o = \dot{S}\,2nA$ & $T\,2nA = o$, atque $T(2n+1)A = \infty$ fore. Similiter inveniuntur $F3\varphi$ (seu $T3\varphi$, $S3\varphi$), $F4\varphi$ &c. seu functiones multiplorum areæ; inter quas ex Formulis 1) & 2) (posito $\psi = n\varphi$) hæ valent relationes:

$$F(n+1)\varphi + F(n-1)\varphi = \frac{2\,Fn\varphi\,.\,F^1\varphi\,.\,F^1o}{(F^1o)^2 \mp (Fn\varphi)^2\,.\,F\varphi^2}$$

& $F(n+1)\varphi\,.\,F(n-1)\varphi = \dfrac{(Fn\varphi^2 - F\varphi^2)\,.\,(F^1o)^2}{(F^1o)^2 \mp Fn\varphi^2\,F\varphi^2}$, atque his subjecta:

$\frac{F\overline{n+1}\,\varphi \cdot F\overline{n-1}\,\varphi}{F\overline{n+1}\,\varphi + F\overline{n-1}\,\varphi} = \frac{(F\overline{n\varphi})^2 - F\varphi^2}{2F_{n\varphi}\cdot F^1\varphi} \cdot F^1 o.$ Ex quo videmus, nostras functiones hujusmodi æquationibus ad differentias satisfacere

$$y_{n+1} + y_{n-1} = \frac{C \cdot y_n}{D \mp y_n^2},\quad y_{n+1}\cdot y_{n-1} = \frac{(y_n^2 - c^2)\,a}{D \mp y_n^2},$$

& $\frac{y_{n+1}\cdot y_{n-1}}{y_{n+1}+y_{n-1}} = \frac{(y_n^2 - c^2)\,a}{C y_n}$; quæ, si universim summari possent, etiam has functiones exhiberent.

Hujusmodi vero consequentias jam ulterius non prosequimur, cum *reliquæ etiam functiones* nobis cordi sint. De eo enim jam agitur, ut & reliquæ functiones secundum varios sectores inter se comparentur. Quem in finem theorema Cel. Eulerianum aliquantum aliter exprimamus, nempe ita: Si fuerit $X = \sqrt{\alpha(\alpha + Cx^2 + \beta x^4)} = '(x)$ & similiter $Y = '(y)$, atque $Z = '(z)$, & præterea $z = \frac{xY + yX}{\alpha - \beta x^2 y^2}$, erit

$$\int_0^x \frac{dx}{X} + \int_0^y \frac{dy}{Y} = \int_0^z \frac{dz}{Z};$$

id, quod ex formula ipsius $F(\varphi + \psi)$, ponendo $F\varphi = x$, $F\psi = y$, $F\overline{\varphi + \psi} = z$, $F^1\varphi = X$ &c., ideoque inverse

$$\varphi = F^{-1}x = \int_0^x \frac{dx}{X},\quad \psi = \int_0^y \frac{dy}{Y},\ \&\ \varphi + \psi = \int_0^z \frac{dz}{Z},$$

facile probatur.

Videamus igitur, num hujus ope formulas subtangentis & secantium inveniamus. Jam itaque antea (p. 51) invenimus $\int_0^x \frac{dx}{X} = \iota^{\cdot} B + \overline{\mathfrak{T}}x$, existente $X^2 = (a^2 - x^2)\cdot(x^2 - b^2) = -a^2 b^2 + \overline{a^2 + b^2}\,x^2 - x^4$, seu $\alpha = \iota^{\cdot} ab$, atque $\beta = \frac{\iota^{\cdot}}{ab}$. Ponamus igitur $x = \mathfrak{T}\varphi$, $y = \mathfrak{T}\psi$, eritque

$\varphi = \overline{\mathfrak{T}}x$, $\psi = \overline{\mathfrak{T}}y$, ideoque $\int\limits_0^x \frac{dx}{X} = \iota \cdot B + \varphi$, atque $\int\limits_0^z \frac{dz}{Z} = \iota \cdot B + \overline{\mathfrak{T}}z =$ $= (\iota \cdot B + \varphi) + (\iota \cdot B + \psi)$, ideoque $\overline{\mathfrak{T}}z = \iota \cdot B + \varphi + \psi$, & igitur $\mathfrak{T}(\iota \cdot B + \varphi + \psi) = z$. Posuimus vero etiam $z = \frac{xY + yX}{\alpha - \beta(xy)^2} = \frac{\mathfrak{T}\varphi \cdot '(\mathfrak{T}\psi) + \mathfrak{T}\psi \cdot '(\mathfrak{T}\varphi)}{\iota \cdot ab - \frac{\iota}{ab} \cdot \mathfrak{T}\varphi^2 \cdot \mathfrak{T}\psi^2}$, exsistente $'(\mathfrak{T}\varphi) = \sqrt{(a^2 - \mathfrak{T}\varphi^2)(\mathfrak{T}\varphi^2 - b^2)} = \mathfrak{d}\mathfrak{T}\varphi = \mathfrak{T}^1\varphi$. Erit igitur $\iota \cdot \mathfrak{T}(\iota \cdot B + \varphi + \psi) = \frac{\mathfrak{T}\varphi \cdot \mathfrak{T}^1\psi + \mathfrak{T}\psi \cdot \mathfrak{T}^1\psi}{ab - \frac{1}{ab}\mathfrak{T}\varphi^2 \cdot \mathfrak{T}\psi^2}$, cujus tamen æquationis sinistra pars sub forma imaginaria exstat, & in $= T(\varphi + \psi)$ mutatur, quo exsultat $T(\varphi + \psi) = \frac{\mathfrak{T}\varphi \cdot \mathfrak{T}^1\psi + \mathfrak{T}\psi \cdot \mathfrak{T}^1\varphi}{ab - \frac{1}{ab} \cdot \mathfrak{T}\varphi^2 \cdot \mathfrak{T}\psi^2}$, quæ formula etiam ex antecedente ipsius $T\overbrace{\varphi + \psi}$ valore, ponendo $\varphi + \iota \cdot B$ pro φ & $\psi - \iota \cdot B$ pro ψ, atque $\iota \cdot \mathfrak{T}\varphi$ pro $T(\varphi \pm \iota \cdot B)$, oritur *). Non igitur ita formulam, quæ $\mathfrak{T}(\alpha + \beta)$ explicaret, realem invenimus, sed imaginariam, quæ in novam pro $T(\varphi + \psi)$ mutatur. Et simile quid ad secantium formulas, quæ hac via invenirentur, observare licet. Similiter enim, quia $\int\limits_0^x \frac{dx}{X} =$ $= A + \iota \cdot B + \mathfrak{S}x$, quando $X^2 = (x^2 - a^2)(x^2 - c^2) = (ac)^2 - (a^2 + c^2)x^2 + x^4$, ideoque $\alpha = ac$, $\beta = \frac{1}{ac}$, invenitur

$$\mathfrak{S}(A + \iota \cdot B + \varphi + \psi) = \frac{\mathfrak{S}\varphi \cdot \mathfrak{S}^1\psi + \mathfrak{S}\psi \cdot \mathfrak{S}^1\varphi}{ac - \frac{1}{ac} \cdot \mathfrak{S}\varphi^2\,\mathfrak{S}\psi^2},$$ exsistente

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*) Ipsa et aliquatenus comprobatur, ponendo $\varphi = o$, ideoque $\mathfrak{T}\varphi = b$, $\mathfrak{T}^1\varphi = o$, quo facto, æquationibus algebraicis olim inventis, convenienter, $T\psi = a \cdot \sqrt{\frac{\overline{\mathfrak{T}}\varphi^2 - b^2}{a^2 - \overline{\mathfrak{T}}\varphi^2}}$, suppeditat.